NOETHERIAN MODULES AND SHORT EXACT SEQUENCES

COMMUTATIVE ALGEBRA

**Proposition.** Let \( 0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0 \) be a short exact sequence of \( A \)-modules. Then,

\[
M \text{ is Noetherian } \iff L \text{ and } N \text{ are Noetherian}
\]

**Proof.** (\( \Rightarrow \)) Given an ascending chain of submodules \( \{L_i\}_{i=1}^{\infty} \) in \( L \), we get ascending chain of submodules \( \{\alpha(L_i)\}_{i=1}^{\infty} \) in \( M \). Since \( M \) is Noetherian, there exists a positive integer \( n \) such that \( \alpha(L_n) = \alpha(L_{n+1}) = \cdots \). Applying \( \alpha^{-1} \) to both sides

\[
\alpha^{-1}(\alpha(L_n)) = \alpha^{-1}(\alpha(L_{n+1})) = \alpha^{-1}(\alpha(L_{n+2})) = \cdots
\]

Since \( \alpha \) is injective, \( \alpha^{-1}(\alpha(L_i)) = L_i \) for each \( i \). So we obtain

\[
L_n = L_{n+1} = L_{n+2} = \cdots
\]

showing that \( L \) is Noetherian.

Similarly, given an ascending chain of submodules \( \{N_i\}_{i=1}^{\infty} \) in \( N \), we get ascending chain of submodules \( \{\beta^{-1}(L_i)\}_{i=1}^{\infty} \) in \( M \). Since \( M \) is Noetherian, there exists a positive integer \( p \) such that \( \beta^{-1}(N_p) = \beta^{-1}(N_{p+1}) = \cdots \). Applying \( \beta \) to both sides,

\[
\beta(\beta^{-1}(N_p)) = \beta(\beta^{-1}(N_{p+1})) = \beta(\beta^{-1}(N_{p+2})) = \cdots
\]

Since \( \beta \) is surjective, \( \beta(\beta^{-1}(N_i)) = N_i \) for each \( i \). So we obtain

\[
N_p = N_{p+1} = N_{p+2} = \cdots
\]

showing that \( N \) is Noetherian.

(\( \Leftarrow \)) Suppose \( \{M_i\}_{i=1}^{\infty} \) is an ascending chain of submodules of \( M \), then identifying \( \alpha(L) \) with \( L \) (which can be done, since \( \alpha \) is injective), and taking intersections, we get a chain of the form

\[
L \cap M_1 \subseteq L \cap M_2 \subseteq \cdots
\]

of submodules in \( L \). Similarly, applying \( \beta \) gives a chain

\[
\beta(M_1) \subseteq \beta(M_2) \subseteq \cdots
\]

of submodules in \( N \). Since \( L \) and \( N \) are Noetherian, each of these chains terminate. To prove that \( M \) is Noetherian, it suffices to prove the following lemma:

**Lemma.** For submodules, \( M_1 \subseteq M_2 \subseteq M \),

\[
\alpha(L) \cap M_1 = \alpha(L) \cap M_2 \quad \text{and} \quad \beta(M_1) = \beta(M_2) \implies M_1 = M_2
\]

**Proof.** Suppose \( m \in M_1 \). Then, \( \beta(m) \in \beta(M_1) = \beta(M_2) \), so that there is \( n \in M_2 \) such that \( \beta(m) = \beta(n) \). Then, \( \beta(m-n) = 0 \), so that \( m-n \in M_1 \cap \ker(\beta) = M_1 \cap \alpha(L) \). It follows that \( m-n \in M_2 \), so that \( m \in M_2 \). This shows that \( M_1 \subseteq M_2 \). Similarly, we can prove \( M_2 \subseteq M_1 \). Thus, \( M_1 = M_2 \), as desired.

\( \square \)

\( \square \)